Lecture 4: Relation, Equivalence relation

Binary Relation: Let A and B be non-empty sets. A binary relation or simply a relation R from A to B is a subset of $A \times B$, that is, $R \subseteq A \times B$.

- If $(a, b) \in R$, then we also say that a is related to b by R or $a\,R\,b$.
- If $A = B$, then we say that R is a relation on A.
- The domain of $R = \{a \in A\ :\ a\,R\,b$ for some $b \in B\}$.
- The range of $R = \{b \in B\ :\ a\,R\,b$ for some $a \in A\}$.
- Let A, B be a sets with $|A| = m$ and $|B| = n$. Then there are 2^{mn} relations from A to B.

Examples:

1. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Since $R \subseteq A \times B$, R is a relation from A to B. The domain of R is $\{1, 3\}$ and the range of R is $\{y, z\}$.
2. Let S be a collection of sets. Then set inclusion $\subseteq$ is a relation on A.
3. The divisibility of two numbers in $\mathbb{N}$ is a relation on $\mathbb{N}$.
4. Let L be the set of lines in the plane. Then perpendicularity of two lines l_1 and l_2 in the plane gives a relation on L.

Complement of relation: Let R be a relation from A to B. The complement of R, denoted by R^c, is a relation from A to B such that $R^c = \{(a, b)\ :\ (a, b) \notin R\}$.

Inverse of relation: Let R be a relation from A to B. The inverse of R, denoted by R^{-1} is a relation from B to A such that $R^{-1} = \{(b, a)\ :\ (a, b) \in R\}$.

Composition of relation: Let A, B and C be sets, and let R be a relation from A to B and S be a relation from B to C, that is, $R \subseteq A \times B$ and $S \subseteq B \times C$. Then

$R \circ S = \{(a, c)\ :\ \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$.

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$. Let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$ be relations from A to B and from B to C respectively. Then $R \circ S = \{(2, z), (3, x), (3, z)\}$.

Types of relation: Let A be a set and R be a relation on A.

1. **Reflexive Relation:** R is reflexive if $(a, a) \in R$, that is, aRa for all $a \in A$.
2. **Symmetric Relation:** R is symmetrix if aRb then bRa.
3. **Antisymmetric Relation:** R is called antisymmetric if aRb and bRa then $a = b$.
4. **Transitive Relation:** R is called transitive: if aRb and bRc then aRc.

Example. Let $A = \{1, 2, 3, 4\}$. Consider the following relations on A.

$R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$,

$R_3 = \{(1, 3), (2, 1)\}$,

$R_4 = \emptyset$, the empty relation,

$R_5 = A \times A$.

Determine, which of the relations are: (a) reflexive, (b) symmetric, (c) antisymmetric, (d) transitive.

Solution: Since $(2, 2) \notin R_1, R_3, R_4$. Hence, these relations are not reflexive. Since $(a, a) \in R_2, R_5$ for every $a \in A$, R_2 and R_5 are reflexive.
R_1 is not symmetric since $(1, 2) \in R_1$ but $(2, 1) \notin R_1$. Similarly R_3 is not symmetric. All other relations are symmetric.
R_2 is not antisymmetric since $(1, 2), (2, 1) \in R_2$ but $1 \neq 2$. Similarly R_5. All the other relations are antisymmetric.
R_3 is not transitive since $(2, 1), (1, 3) \in R_3$ but $(2, 3) \notin R_3$. All the other relations are transitive.

Equivalence Relation: A relation R on a set S is called an equivalence relation if it is reflexive, symmetric, and transitive.

Examples:

1. Let S be a set of lines in the plane. The relation of parallel is an equivalence relation.
2. The relation of inclusion $\subseteq$ is not equivalence relation. It is reflexive and transitive but not symmetric, since $A \subseteq B$ does not imply $B \subseteq A$.
3. Let m be a fixed positive integer. Two integers a and b are said to be congruent moulo m, written as $a \equiv b \pmod{m}$, if m divides $a - b$. This relation of congruence modulo m is an equivalence relation on $\mathbb{Z}$.

Equivalence Class: Let R be an equivalence relation on a set S. For $a \in S$, the set $[a] = \{x : (a, x) \in R\}$ is called the equivalence class of a.

The collection of all such equivalence classes is denoted by S/R, that is, $S/R = \{[a] : a \in S\}$. The set S/R is also called quotient set of S by R.

Example In the above Example 3, the relation of congruent modulo m on the set of integers $\mathbb{Z}$. Let $m = 5$. Then we see that

$[0] = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$, that is, $[0] = \{5k : k \in \mathbb{Z}\}$,

$[1] = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$, that is, $[1] = \{5k + 1 : k \in \mathbb{Z}\}$,

$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$, that is, $[2] = \{5k + 2 : k \in \mathbb{Z}\}$.

$[3] = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$, that is, $[3] = \{5k + 3 : k \in \mathbb{Z}\}$.

$[4] = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$, that is, $[3] = \{5k + 4 \,:\, k \in \mathbb{Z}\}$.

The above are the only distinct equivalence classes. Thus $\mathbb{Z}/R = \{[0], [1], [2], [3], [4]\}$.

Theorem 1: Let R be an equivalence relation on a set S.

1. For each $a \in S$, $a \in [a]$, that is, every element lies in its own equivalence class.
2. For each $a, b \in S$, $a\,R\,b$ if and only if $[a] = [b]$, that is, if any two elements are related by R then they have same equivalence class.
3. For each $a, b \in S$, $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Proof: Since R is reflexive, $a\,R\,a$ for each $a \in S$. So $a \in [a]$. This proves first part.

Second Part: Suppose $a\,R\,b$ and $x \in [a]$. Then $x\,R\,a$. Since $a\,R\,b$ and R is transitive, $x\,R\,b$. So $x \in [a]$, and $[a] \subseteq [b]$. Similarly we see that $[b] \subseteq [a]$. Combining both, we get $[a] = [b]$. Conversely, let $[a] = [b]$. This means, if $x \in [a]$ then $x \in [b]$ and therefore xRa (or aRx since R is symmetrix) and xRb. Since R is transitive, aRb.

Third Part: let $[a] \cap [b] \neq \emptyset$ and $x \in [a] \cap [b]$. Then xRa (so aRx) and xRb imples aRb. By second part, $[a] = [b]$.

Partition of a set: Let S be a non-empty set. A collection P, containing subsets $A_1, A_2, \ldots$ of S, is called a partition of S if: $\cup A_i = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Example: Let $S = \{1, 2, \ldots, 9\}$. Consider the following collections of subsets of S.

$P_1 = [\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}]$,

$P_2 = [\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}]$,

$P_3 = [\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}]$.

P_1 is not a partition, since $7 \notin P_1$. P_2 is not a partition, since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. Note that P_3 is a partition of S.

Theorem 2: Let R be an equivalence relation on a nonempty set S. The collection S/R of all equivalence classes gives a partition of S.

Proof: Proof follows from Theorem 1.

Counting partitions: The total number of partitions of an n-element set is the Bell number B_n. The first several Bell numbers are $B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52$, and $B_6 = 203$. Bell numbers satisfy the recurrence relation (we will teach recurrence relation in coming lectures) involving binomial coefficients: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$

Hint: It can be explained by observing that, from an arbitrary partition of n + 1 items, removing the set containing the first item leaves a partition of a smaller set of k items for some number k that may range from 0 to n. There are $\binom{n}{k}$ choices for the k items that remain after one set is removed, and B_k choices of how to partition them.

Bell triangle: The Bell numbers may also be computed using the Bell triangle in which the first value in each row is copied from the end of the previous row, and subsequent values are computed by adding two numbers, the number to the left and the number to the above left of the position.

Here are the first five rows of the triangle constructed by the above rules:

1
1 2
2 3 5
5 7 10 15
15 20 27 37 52
The Bell numbers appear on both the left and right sides of the triangle

Applications in Counting the Equivalence relations and Factorizations:

- Recall the number of ways of partitioning a finite set into subsets is equal to the number of equivalence relations on the set.
- If a number N is a square-free positive integer, (meaning that it is the product of some number n of distinct prime numbers), then B_n gives the number of different multiplicative partitions of N. These are factors of N into numbers greater than one, treating two factorizations as the same if they have the same factors in a different order. For instance, 30 is the product of the three primes $2, 3$, and 5, and has $B_3 = 5$ factorizations:
$$30 = 2 \times 15 = 3 \times 10 = 5 \times 6 = 2 \times 3 \times 5.$$